B.Sc. Part II (Hons) 4th Paper

Vector calculus

Q. Prove that $\nabla^2 (r^n \vec{r}) = n(n+3) r^{n-2} \vec{r}$

Soln where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

Let $\vec{F} = r^n \vec{r}$

$$\Rightarrow \vec{F} = r^n (x\vec{i} + y\vec{j} + z\vec{k})$$

$$\Rightarrow \vec{F} = r^n x \vec{i} + r^n y \vec{j} + r^n z \vec{k} \quad \text{—(1)}$$

Let $f_1 = r^n x,\ f_2 = r^n y,\ f_3 = r^n z$ —(2)

So, using (2) in (1), we get

$$\vec{F} = f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}$$

Now $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\therefore \text{LHS} = \nabla^2 (r^n \vec{r}) = \nabla^2 \vec{F}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \vec{F}$$

$$= \frac{\partial^2 \vec{F}}{\partial x^2} + \frac{\partial^2 \vec{F}}{\partial y^2} + \frac{\partial^2 \vec{F}}{\partial z^2}$$

$$= \frac{\partial^2}{\partial x^2}(f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}) + \frac{\partial^2}{\partial y^2}(f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}) + \frac{\partial^2}{\partial z^2}(f_1 \vec{i} + f_2 \vec{j} + f_3 \vec{k}) \quad \text{—(3)}$$

$$= \left(\frac{\partial^2}{\partial x^2} f_1 + \frac{\partial^2}{\partial y^2} f_1 + \frac{\partial^2}{\partial z^2} f_1 \right) \vec{i} + \left(\frac{\partial^2}{\partial x^2} f_2 + \frac{\partial^2}{\partial y^2} f_2 + \frac{\partial^2}{\partial z^2} \right) \vec{j} + \left(\frac{\partial^2}{\partial x^2} f_3 + \frac{\partial^2}{\partial y^2} f_3 + \frac{\partial^2}{\partial z^2} f_3 \right.$$

∴ ~~LHS = $\frac{\partial^2}{\partial x^2}$~~

Now, $r^2 = x^2+y^2+z^2$

$\Rightarrow 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$

similarly $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$ — (4)

Now, $f_1 = r^n x$

$$\Rightarrow \frac{\partial f_1}{\partial x} = \frac{\partial}{\partial x}(r^n x) = \frac{\partial (r^n)}{\partial x}\cdot x + r^n \cdot \frac{\partial x}{\partial x}$$

$$\Rightarrow \frac{\partial f_1}{\partial x} = n r^{n-1}\frac{\partial r}{\partial x}\cdot x + r^n = n\cdot r^{n-1}\cdot x\cdot\frac{x}{r} + r^n \quad [\text{using } (4)]$$

$$\Rightarrow \frac{\partial f_1}{\partial x} = n r^{n-2}x^2 + r^n$$

~~Similarly~~ $$\Rightarrow \frac{\partial^2 f_1}{\partial x^2} = \frac{\partial}{\partial x}(n r^{n-2}x^2 + r^n)$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial x^2} = n x^2\cdot(n-2)r^{n-3}\cdot\frac{\partial r}{\partial x} + n r^{n-2}\cdot 2x + n r^{n-1}\frac{\partial r}{\partial x}$$

$$= n x^2\cdot(n-2)r^{n-3}\cdot\frac{x}{r} + 2nx r^{n-2} + n r^{n-1}\cdot\frac{x}{r}$$

$$= n(n-2)x^3 r^{n-4} + 2nx r^{n-2} + nx r^{n-2}$$

$$= 3nx r^{n-2} + n(n-2)x^3 r^{n-4}$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial x^2} = nx r^{n-2}\left[3 + \frac{(n-2)x^2}{r^2}\right] \quad \text{— (5)}$$

Now, $f_1 = r^n x$

$$\Rightarrow \frac{\partial f_1}{\partial y} = n r^{n-1} \frac{\partial r}{\partial y} \cdot x + r^n \cdot 0 \quad = n r^{n-1} \cdot \frac{y}{r} \cdot x$$

$$\Rightarrow \frac{\partial f_1}{\partial y} = n x y r^{n-2}$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial y^2} = \frac{\partial}{\partial y}\left[n x y r^{n-2}\right] = n x \left[r^{n-2} + y (n-2) r^{n-3} \cdot \frac{y}{r}\right]$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial y^2} = n x \left[r^{n-2} + (n-2) y^2 r^{n-4}\right]$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial y^2} = n x r^{n-2}\left[1 + (n-2)\frac{y^2}{r^2}\right] \quad \text{——} (6)$$

Again, $\frac{\partial f_1}{\partial z} = \frac{\partial}{\partial z}(r^n x) = x \cdot n r^{n-1} \cdot \frac{\partial r}{\partial z}$

$$\Rightarrow \frac{\partial f_1}{\partial z} = n x r^{n-1} \cdot \frac{z}{r} = n x z r^{n-2}$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial z^2} = n x \frac{\partial}{\partial z}\left(z r^{n-2}\right) = n x \left[r^{n-2} + z (n-2) r^{n-3} \cdot \frac{z}{r}\right]$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial z^2} = n x r^{n-2}\left[1 + (n-2)\frac{z^2}{r^2}\right] \quad \text{——} (7)$$

Adding (5), (6) and (7), we get

$$\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}$$

$$= n x r^{n-2}\left[3 + \frac{(n-2)x^2}{r^2} + 1 + \frac{(n-2)y^2}{r^2} + 1 + (n-2)\frac{z^2}{r^2}\right]$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} = n x r^{n-2}\left[5 + \frac{n-2}{r^2}(x^2+y^2+z^2)\right]$$

$$= n x r^{n-2}\left[5 + \frac{n-2}{r^2}\cdot r^2\right]$$

$$\Rightarrow \frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2} = n(n+3)\, x\, r^{n-2} \quad \text{---(8)}$$

Similarly,

$$\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2} = n(n+3)\, y\, r^{n-2} \quad \text{---(9)}$$

$$\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2} = n(n+3)\, z\, r^{n-2} \quad \text{---(10)}$$

Putting these values from (8), (9) and (10) in (3), we get

$$LHS = n(n+3)\left[x\vec{i} + y\vec{j} + z\vec{k}\right] r^{n-2}$$

$$= n(n+3)\, r^{n-2}\, \vec{r} = RHS$$